MINIMAL BOUNDED INDEX SUBGROUP FOR DEPENDENT THEORIES

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Abstract. For a dependent theory T, in $\mathfrak{C}_T$ for every type definable group G, the intersection of type definable subgroups with bounded index is a type definable subgroup with bounded index.

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§0 Introduction

Assume that T is a dependent (complete first order) theory $\mathfrak{C}$ is a $\bar{\kappa}$-saturated model of T (a monster) G is a type definable (in $\mathfrak{C}$) group in $\mathfrak{C}$ (of course we consider only types of cardinality $< \bar{\kappa}$).

A type definable subgroup H of G is call bounded if the index $(G : H)$ is $< \bar{\kappa}$. We prove that there is a minimal bounded definable subgroup. The first theorem on this line for T stable is of Baldwin-Saxe [BaSx76].

Recently Hrushovski, Peterzil and Pillay [HPP0x] investigated definable groups, 0-minimality and measure, see also the earlier work on definable subgroups in 0-minimal T in Bezarducci, Otero, Peterzil and Pillay.

Hrushovski in a lecture in the logic seminar in the Hebrew University mentioned that "for every definable group in $\mathfrak{C}_T$, the intersection of the definable subgroup with bounded index is of bounded index" is proved in [HPP0x] for definable groups with suitable definable measure on them and T dependent; but it is not clear if the existence of definable measure is necessary.

Recent works of the author on dependent theories are [Sh 783] (see §3,§4 on groups) [Sh 863] (e.g. the first order theory of the peadics is strongly^1 dependent but not strongly^2 dependent, see end of §1; on strongly^2 dependent fields see §5) and [Sh:F705]. This work is continued in [Sh:F753] (for G abelian and $\mathbb{L}_{\infty,\bar{\kappa}}$-definable subgroups).

§1

1.1 Lemma. *For T dependent.*
1) If $\circledast$ below holds <u>then</u>:

- (α) *$q(\mathfrak{C})$ is a subgroup of $p(\mathfrak{C})$*
- (β) *$q(\mathfrak{C})$ is of index $\leq 2^{|T|^{\aleph_0}}$*
- (γ) *essentially $q(x) \backslash p(x)$ is of cardinality $\leq |T|^{\aleph_0}$ (i.e., for some $q'(x) \subseteq q(x)$ of cardinality $\leq |T|^{\aleph_0}, q(x)$ is equivalent to $p(x) \cup q'(x)$).*

<u>*where*</u>

$\circledast$ (a) *$p(x)$ is a type such that $p(\mathfrak{C})$ is a group we call G (under some definable operation xy, x^{-1} and the identity e_G which are constant here; of course all types are of cardinality $< \bar{\kappa}, \mathfrak{C}$ is $\bar{\kappa}$-saturated)*

(b) *$q(x) = p(x) \cup \bigcup\{r(x) : r(x) \in \mathbf{R}\}$ where*

(c) *$\mathbf{R} = \{r(x) : r(x)$ a type such that $(p \cup r)(\mathfrak{C})$ is a sub-group of $p(\mathfrak{C})$ of index $< \bar{\kappa}\}$.*

2) There exists some $q' \subseteq q$ over $Dom(p)$, equivalent to q and such that $|q'| \leq |T| + |Dom(p)|$. So $(p(\mathfrak{C}) : q(\mathfrak{C})) \leq 2^{|\mathrm{Dom}(p)|+|T|}$.
3) If $r_i(x) \in \mathbf{R}$ for $i < (|T|^{\aleph_0})^+$ then for some $\alpha < (|T|^{\aleph_0})^+$ we have $(p(x) \cup \bigcup\{r_i(x) : i < \alpha\})(\mathfrak{C}) = (p(x) \cup \bigcup\{r_i(x) : i < (|T|^{\aleph_0})^+\})(\mathfrak{C})$.

Proof. 1) Note

$\circledast_1$ $\mathbf{R}$ is closed under unions of $< \bar{\kappa}$ hence $q' \subseteq q \wedge |q'| < \bar{\kappa} \Rightarrow q' \in \mathbf{R}$

$\circledast_2$ if $r(x) \in \mathbf{R}, r'(x) \subseteq r(x)$ is countable <u>then</u> there is a countable $r''(x) \subseteq r(x)$ including $r'(x)$ which belongs to $\mathbf{R}$
[Why? Let $r'(x) = \{\varphi_n(x, \bar{a}_n) : n < \omega\}$ (can use $\varphi_n = (x = x)$). Without loss of generality $r(x)$ is closed under conjuctions and also $r'(x)$. Now we choose $\psi_n(x, \bar{b}_n) = \psi^1_n(x, \bar{b}^1_n) \wedge \psi^2_n(x, \bar{b}^2_n)$ with $\psi^1_n(x, \bar{b}^1_n) \in r(x), \psi^2_n(x, \bar{b}^2_n) \in p(x)$ by induction on $n < \omega$ such that $\psi_{n+1}(x, \bar{b}_{n+1}) \wedge \psi_{n+1}(y, \bar{b}_{n+1}) \vdash \psi_n(xy^{-1}, \bar{b}_n) \wedge \varphi_n(x, \bar{a}_n)$. Such formula exists as $(p(x) \cup r(x)) \cup (p(y) \cup r(y)) \vdash \psi_n(xy^{-1}, \bar{b}_n) \wedge \varphi_n(x, \bar{a}_n)$.
Now $r''(x) = \{\varphi_n(x, \bar{a}_n), \psi_n(x, \bar{b}_n) : n < \omega\}$ is as required.]

Clause (α) is obvious.
Assume toward contradiction that the conclusion (β) fails. So we can choose (c_α, r_α) by induction on $\alpha < (|T|^{\aleph_0})^+$ such that

$\circledast_3$ (a) $c_\alpha \in (p(x) \cup \bigcup\{r_\beta : \beta < \alpha\})(\mathfrak{C})\backslash q(\mathfrak{C})$

(b) $r_\alpha = \{\psi^\alpha_n(x, \bar{b}^\alpha_n) : n < \omega\} \subseteq q$ and $\bar{b}^\alpha_n \triangleleft \bar{b}^\alpha_{n+1}$

(c) $\psi^\alpha_{n+1}(x, \bar{b}^\alpha_{n+1}) \vdash \psi^\alpha_n(x, \bar{b}^\alpha_n)$

(d) $r_\alpha \in \mathbf{R}$

(e) c_α does not realize r_α in fact $\mathfrak{C} \models \neg\psi^\alpha_0(c_\alpha, \bar{b}^\alpha_0)$.

Without loss of generality

$\circledast_4$ $\psi^\alpha_{n+1}(x, \bar{y}^\alpha_{n+1}) \vdash \psi^\alpha_n(x, \bar{y}^\alpha_n)$ and $(\psi^\alpha_{n+1}(x_1, \bar{y}^\alpha_{n+1}) \wedge \psi^\alpha_{n+1}(x_2, \bar{y}^\alpha_{n+1})) \rightarrow (\psi^\alpha_{n+1}(e, \bar{y}^\alpha_{n+1}) \wedge \psi^\alpha_{n+1}(x_1 x_2^{-1}, y^\alpha_{n+1}))$

$\circledast_5$ $\psi^\alpha_n(x, \bar{y}^\alpha_n) = \psi_n(x, \bar{y}_n)$ and $\psi_{n+1}(x, \bar{y}_{n+1})) \vdash \psi_n(x, \bar{y}_n)$.

$\circledast_6$ $\langle c_\alpha \bar{\mathbf{a}}_\alpha : \alpha < (|T|^{\aleph_0})^+$ is an indiscernible sequence over $\mathrm{Dom}(p)$ where $\bar{\mathbf{a}}_\alpha = \bar{b}^\alpha_0 {}^\frown \bar{b}^\alpha_1 {}^\frown \bar{b}^\alpha_2 {}^\frown \ldots$, without loss of generality $\bar{b}^\alpha_n = \bar{\mathbf{a}}_\alpha \restriction k_n$
[Why? By Ramsey theorem and compactness.]

$\circledast_7$ if $\alpha < \beta < \gamma$ then $c_\alpha c_\beta^{-1} \in r_\gamma(\mathfrak{C})$.
[Why? Without loss of generality γ is infinite, as $(p \cup r_\gamma)(\mathfrak{C})$ is a subgroup of $p(\mathfrak{C})$ of index $< \bar{\kappa}$. If $\gamma \geq \omega, \langle c_i : i < \gamma\rangle$ is a sequence of indiscernibles over $\mathrm{Dom}(p) \cup \bar{b}_\gamma$ of elements of $p(\mathfrak{C})$ pairwise non-equivalent modulo $G_\gamma = (p \cup r_\gamma)(\mathfrak{C})$, then extend it to $\langle c_i : i < \bar{\kappa}\rangle$ a sequence of indiscernibles over $\mathrm{Dom}(p) \cup \bar{b}_\gamma$ and arrive at $\alpha < \beta \Rightarrow c_\alpha c_\beta^{-1} \notin G_\gamma \Rightarrow c_\beta c_\alpha^{-1} \notin G_\gamma$ so $\langle c_\alpha G_\gamma : \alpha < \bar{\kappa}\rangle$ pairwise distinct (equivalently $\langle G_\gamma c_\alpha : \alpha < \bar{\kappa}\rangle$ pairwise distinct) contradiction.]

$\circledast_8$ $c_\alpha \in r_\beta(\mathfrak{C})$ iff $\alpha \neq \beta$.
[Why? Let

$$c^*_\alpha = c_{2\alpha+1} \cdot (c_{2\alpha})^{-1}$$

$$r^*_\alpha = r_{2\alpha}.$$

So:

(i) if $\beta < \alpha, c^*_\alpha \in (p \cup r^*_\beta)(\mathfrak{C})$ as $c_{2\alpha+1}, c_{2\alpha}$ belong to the subgroup $(p \cup r_{2\beta+1})(\mathfrak{C})$ by clause (a) of $\circledast_3$

(ii) if $\beta > \alpha, c^*_\alpha$ belongs to $(p \cup r^*_\beta)(\mathfrak{C})$ by $\circledast_7$

(iii) if $\beta = \alpha$ then c^*_α does not belong to $(p \cup r^*_\beta)(\mathfrak{C})$ as it is a subgroup, $c_{2\alpha+1}$ belongs to it and $c_{2\alpha}$ does not belong to it by clause (e) of $\circledast_3$.

Let $\bar{\mathbf{a}}^*_\alpha = \bar{\mathbf{a}}_{2\alpha+1}, \bar{b}^{\alpha^*}_n = \bar{b}^{2\alpha}_n$ retaining the same ψ's. So we have gotten an example as required in $\circledast_8$ (not losing the other demands).]

$\circledast_9$ if $d_1, d_2 \in (p \cup r_\alpha)(\mathfrak{C})$ then $d_1 c_\alpha d_2 \notin \varphi_1(\mathfrak{C}, \bar{b}^\alpha_1)$.
[Why? Fix α, if this holds for some $\varphi_n(-, \bar{b}^\alpha_n)$ by indiscernibility renaming the φ_i's this is O.K. Otherwise for each $n < \omega$ there are $d^n_1, d^n_2 \in (p \cup r_\alpha)(\mathfrak{C})$ such that $\mathfrak{C} \models \varphi_n(d^n_1 c_\alpha d^n_2, \bar{b}^\alpha_n)$. By compactness for some $d^*_1, d^*_2 \in (p \cup r_\alpha)(\mathfrak{C})$ we have $\models \varphi_n[d^*_1 c_\alpha d^*_2, \bar{b}^\alpha_n]$ for every $n < \omega$. So $d^*_1 c_\alpha d^*_2$ belongs to the subgroup $(p \cup r_\alpha)(\mathfrak{C})$ but also d^*_1, d^*_2 belongs to it hence c_α belongs, contradiction.]

$\circledast_{10}$ if $w = \{i_1, \dots, i_n\}, i_1 < \dots < i_n < (2^{|T|})^+$, and $d_w := c_{i_1} c_{i_2} \dots c_{i_n}$ then $\models \varphi_1[d_w, \bar{b}^1_\alpha] \Leftrightarrow \alpha \notin w$.
[Why? If $\alpha \in w$ let k be such that $\alpha = i_k$, so $(c_{i_1}, \dots, c_{i_{k-1}}) \in (p \cup r_\alpha)(\mathfrak{C})$ by $\circledast_7$ and $c_{i_{k+1}} \dots c_{i_n} \in (p \cup r_\alpha)(\mathfrak{C})$ hence $d_w = (c_{i_1} \dots c_{i_k}) c_{i_k} (c_{i_{k+1}} \dots c_{i_n}) \notin (p \cup r_\alpha)(\mathfrak{C})$ by $\circledast_9$.
Second, if $\alpha \notin w$ by $\circledast_8$ as $\{c_{i_\ell} : \ell < n\}$ is included in the subgroup $(p \cup r_\alpha)(\mathfrak{C})$.]

So we get a contradiction to "T dependent" hence clause (β) holds. Also clause (γ) follows by the following observation:

Observation. If $r(x) \in \mathbf{R}$ and $|r(x)| \leq \theta$ then $(p(\mathfrak{C}) : (p \cup r)(\mathfrak{C})) \leq 2^\theta$ (except finite when θ is finite).

Proof. 1) If θ is finite then by compactness. If σ is infinite then without loss of generality r is closed under conjunctions. Let $r = \{\varphi_i(x, \bar{\mathbf{b}}) : i < \theta\}, \bar{\mathbf{b}}$ is possibly infinite.

For each $i < \theta$ let $u \subseteq$ ord be such that $\bar{\kappa} > |u| > (p(\mathfrak{C}) : (p \cup r)(\mathfrak{C}))$ let $\Gamma_{i,u} = \cup\{p(x_\alpha) : \alpha \in u\} \cup \{\neg\varphi_i(x_\alpha x^{-1}_\beta, \bar{\mathbf{b}}) : \alpha < \beta \text{ from } u\}$. So for some finite $u^*_i \subseteq u, \Gamma_{i,u^*_i}$ is contradictory so Γ_{i,n_i} is contradictory when $n_i = |u_i|$. It suffices to use $(2^\theta)^+ \rightarrow (\dots n_i \dots)_{i<\theta}$ (why? let $\langle c_\alpha : \alpha < (2^\theta)^+\rangle$ exemplify the failure and let $\zeta_{\alpha,\beta} = \text{Min}\{i : \models \neg\varphi_i(c_\alpha c^{-1}_\beta, \bar{\mathbf{b}})\}$).
2) Observe that every automorphism of $\mathfrak{C}$ fixing $\text{Dom}(p)$ maps $p(\mathfrak{C})$ onto itself and therefore maps $q(\mathfrak{C})$ onto itself.

It follows that if $c_1, c_2 \in p(\mathfrak{C})$ are such that $\text{tp}(c_1, \text{Dom}(p)) = \text{tp}(c_2, \text{Dom}(p))$ then $c_1 \in q(\mathfrak{C})$ if and only if $c_2 \in q(\mathfrak{C})$. Let $\mathbf{P} := \{\text{tp}(b/\text{Dom}(p)) | b \in q(\mathfrak{C})\}, \mathbf{P}(\mathfrak{C}) := \{r(\mathfrak{C}) : r \subset \mathbf{P}\}$. Then by the above explanation $\mathbf{P}(\mathfrak{C}) \subseteq q(\mathfrak{C})$. By definition $q(\mathfrak{C}) \subseteq p(\mathfrak{C})$ so they are equal. Let $q_{**} = \cap\{r : r \in \mathbf{P}\}$ then $q(\mathfrak{C}) \subseteq q_{**}(\mathfrak{C}))$.

If they are equal then we are done. Otherwise take $c_1 \in q_{**}(\mathfrak{C}) \backslash q(\mathfrak{C})$. Without loss of generality let $\psi(x, \bar{d}) \in q$ be such that $\models \neg\psi(c_1, \bar{d})$.

By definition of $\mathbf{P}$ and c_1, for each $\theta(x,\bar{e}) \in \operatorname{tp}(c_1, \operatorname{Dom}(p))$ there exists some $p_{\theta(x,\bar{e})} \in \mathbf{P}$ such that $\sigma(x,\bar{e}) \in p_{\sigma(x,\bar{e})}$ and therefore some $c_{\sigma(x,\bar{e})} \in q(\mathfrak{C})$ realizes $\theta(x,\bar{e})$. So $\operatorname{tp}(c_1, \operatorname{dom}(p)) \cup q(x)$ is finitely satisfiable and is therefore realized by some c_2. Thus $\operatorname{tp}(c_1, \operatorname{Dom}(p)) = \operatorname{tp}(c_2, \operatorname{Dom}(p))$, but $c_1 \notin q(\mathfrak{C})$ and $c_2 \in q(\mathfrak{C})$ a contradiction.

3) By the proof of part (1). $\square_{1.1}$

§2

Claim. *[T dependent] Assume*

- (a) *G is a A^*-definable semi-group with cancellation*
- (b) *$q(x, \bar{a})$ is a type, $q(\mathfrak{C}, \bar{a})$ a sub-semi group.*

Then we can find q^* and $\langle \bar{a}_i : i < \alpha \rangle$ and B such that

- (α) $\alpha < (|T|^{\aleph_0})^+$
- (β) $\mathrm{tp}(\bar{a}_i, A^*) = \mathrm{tp}(\bar{a}, A^*)$
- (γ) $q^* = \cup\{q_i(\bar{x}, \bar{a}_i) : i < \alpha\}$
- (δ) $B \subseteq \cap\{q_i(\mathfrak{C}, \bar{a}_i) : i < \alpha\}$ *and* $|B| \leq |\alpha|$
- (ε) *if* $\bar{a}'$ *realizes* $\mathrm{tp}(\bar{a}, A^*)$ *and* $B \subseteq q(\mathfrak{C}, \bar{a}')$ *then* $q^*(\mathfrak{C}) \subseteq q(\mathfrak{C}, \bar{a}')$.

Proof. We try to choose $\bar{a}_\alpha, b_\alpha$ by induction on $\alpha < (|T|^{+\aleph_0})^+$ such that

$\circledast$ (a) $\bar{a}_\alpha$ realizes $\mathrm{tp}(\bar{a}, A^*)$
- (b) $b_\alpha \notin q(\mathfrak{C}, \bar{a}_\alpha)$
- (c) b_α realizes $q(x, \bar{a}_\beta)$ for $\beta < \alpha$
- (d) b_β realizes $q(x, \bar{a}_\alpha)$ for $\beta < \alpha$.

If we succeed we get contradiction as in the proof in §1. If we are stuck at some $\alpha < (|T|^{\aleph_0})^+$ then take $\langle \bar{a}_i : i < \alpha \rangle, B = \{b_i : i < \alpha\}$.

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